Math 246A Lecture 13 Notes

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1 Integration of Harmonic Functions and Winding Numbers

1.1 Integrating harmonic functions

Recall the definition of a harmonic function.

Definition 1.1. A function $u : \mathbb{R}^2 \to \mathbb{R}$ is **harmonic** if $u_{xx} + u_{yy} = 0$.

Example 1.1. Let $u = \mathrm{Re}(f)$ and $f = u = iv$ be holomorphic. Then, by the Cauchy Riemann equations, u is harmonic.

Assume u is harmonic in Ω. Let $*du = -u_y dx + u_x dy$. Then

$$\int_{\partial R} *du = 0$$

for all rectangles with $\partial R \subseteq \Omega$. This is iff $f \in H(\Omega)$ and $u = \mathrm{Re}(f)$ in Ω, which is equivalent to $\int_\gamma *du = 0$ for all closed curves $\gamma \subseteq \Omega$.

However, if Ω is an annulus and $u = \log|z|$, then $\int_{\partial R} *du \neq 0$. So we must take care when formulating our theorem.

Theorem 1.1. *Let $u(z)$ be harmonic on a domain Ω. Then for $0 < r < R$,*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt.$$

Corollary 1.1. *Let Ω be a domain and $u : \Omega \to \mathbb{R}$ be harmonic. If $u(z_0) = \sup_{B(z_0,R)} u(z)$, then u is constant on Ω.*

1.2 Winding numbers

Let $\gamma = \{z(t) : a \leq t \leq b\}$ be a continuous, closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \gamma$. Then let $\varepsilon > 0$ be such that $\int_\gamma |z - \alpha| > \varepsilon > 0$. Then there exists $\delta > 0$ such that $|t < s| < \delta \implies |\gamma(t) - \gamma(s)| < \varepsilon$. Choose $a = t_0 < t_1 < \cdots < t_n = b$ with $t_{j+1} - t_j < \delta$ for all

j. Let $U_j = \{z \in \mathbb{C} : |z - z(t_j)| < \varepsilon\}$. Then $U_0 = U_n$, $U_j \supseteq \{z(t) : t_{j-1} \le t \le t_{j+1}\}$, and $\alpha \notin \bigcup_j U_j$.

On U_j, there exists a holomorphic function F_j where $F_j'(z) = \frac{1}{z-\alpha}$.

$$F_j(z) = \log|z - \alpha| + i\arg(z - \alpha) + 2\pi i k_j,$$

where $k_j \in \mathbb{Z}$.

Definition 1.2. The **winding number** (or **index of γ around α**) is

$$n(\gamma, \alpha) = \frac{1}{2\pi i}\sum_{j=0}^{n-1} F_j(z(t_{j+1})) - F_j(z(t_j)).$$

Theorem 1.2. *The winding number has the following properties:*

1. $n(\gamma, \alpha) \in \mathbb{Z}$.
2. $n(\gamma, \alpha)$ *is independent of choices of* t_j *and of* F_j
3. *If* γ *is piecewise* C^1, *then* $n(\gamma, \alpha) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-\alpha}\,dz$
4. *The function* $\alpha \mapsto n(\gamma, \alpha)$ *is constant on components of* $\mathbb{C} \setminus \gamma$
5. $n(\gamma, \alpha) = 0$ *on the unbounded componnent of* $\mathbb{C} \setminus \gamma$.

Proof. To prove the first statement,

$$e^{2\pi i n(\gamma,\alpha)} = \prod_{j=1}^{n} e^{F_j(z(t_{j+1})) - F_j(z(t_j))} = \prod_{j=1}^{n} \frac{z(t_{j+1}) - \alpha}{z(t_j) - \alpha} = 1.$$

To prove the second statement, we just need to show that the winding number is unchanged if we take a refinement of the partition. This corresponds to the same F_j.

The equality of winding numbers under refinements of partitions makes the Riemann sum in part 3 the same for sufficiently fine partitions.

The winding number is continuous as a function of α, but it is integer valued. So it is constant on connected components.

In the integral form in part 3, let $\alpha \to \infty$. Then the integrand goes to zero uniformly, so the winding number is zero on this component. □